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Beautiful Gauge Field Equations in Clifforms[1]

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By combining the tetrads of unholonomic frames of spacetime with the Dirac matrices to a one-form γ, we can reformulate not only the Dirac equation, but also the Einstein equations and supergravity in a very concise form. These "Clifforms" also shed some light on the chiral decomposition à la Ashtekar, the role of the *axion* as a dynamical degree of freedom dual to the torsion of the Einstein–Cartan theory, and the role of the Seiberg–Witten equation for S-duality.

1. INTRODUCTION

This paper was inspired by the beautiful paper of Freedman (1993), in which the main ideas and principles of the gauge approach to fundamental physics were highlighted. The formal beauty of these equations is intended here to be enhanced by applying Clifford algebra-valued differential forms, or *Clifforms* for short. These are based on Cartan's exterior calculus in curved spacetime as well as on the Clifford algebra à la Dirac and allow for a rather compact and illuminating notation.

After the advent of the Dirac equation, some years later in 1932, Schrödinger (1932) devised already such a Clifford algebra version of Riemannian geometry. With respect to the matter couplings, these Clifforms allow us to describe rather naturally the couplings of gravity to matter spinor fields, i.e., the Dirac field with spin 1/2 and the Rarita–Schwinger field with spin 3/2 (Mielke *et al.*, 1996a).

The Einstein–Cartan theory coupled to Dirac spinors as well as its extension to *supergravity* (SUGRA) can be completely expressed in Clifforms. The chiral reformulation à la Ashtekar (1986, 1991) is presented here from

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the more fundamental point of view of the generating function (Mielke, 1990b) constructed from the translational Chern–Simons term.

For the Dirac equation, we find an effective nonlinearity induced by torsion, resembling the Heisenberg nonlinear spinor equation. As a *new* development, we contrast this four-fermion-type interaction with a dynamical axion coupling resulting from strings.

Moreover, in the case of Yang–Mills fields coupled to Weyl spinors, we propose here to proceed from a Gordon-decomposed spinor Lagrangian with a (self-dual) interaction to the polarization current. Minimizing solutions turn out to necessarily satisfy the Seiberg–Witten equations (1994). Their role for the index theorem and for the 4D Donaldson invariants has recently been reviewed by Atiyah (1984, 1998).

2. THE CLIFFORD BASIS OF SPACETIME GEOMETRY

Noninertial frames of spacetime will be described by means of a very concise formalism employing Clifford algebra-valued exterior differential forms. We will proceed from a representation (Bjorken and Drell, 1964; Kaku, 1993) by the Dirac matrices γ_α obeying

$$\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2o_{\alpha\beta}\mathbf{1}_4 \tag{1}$$

where $\alpha, \beta = \hat{0}, \hat{1}, \hat{2}, \hat{3}$ denote the (anholonomic) indices of the frame field e_α, which is assumed to be *orthonormal*. The signature of the Minkowski metric $o_{\alpha\beta} = e^i{}_\alpha e^j{}_\beta g_{ij}$ of the frame bundle is $(o_{\alpha\beta}) = diag(1, -1, -1, -1)$. We choose the 16 matrices $\{\mathbf{1}_4, \gamma_\alpha, \sigma_{\alpha\beta}\ \gamma_5, \gamma_5\gamma_\alpha\}$, where $\sigma_{\alpha\beta} := \frac{1}{2}i(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)$ are the Lorentz generators and $\gamma_5 = -i\gamma_{\hat{0}}\gamma_{\hat{1}}\gamma_{\hat{2}}\gamma_{\hat{3}}$, as a basis of the *Clifford algebra* in four dimensions. With respect to the *trace*, the elements of the Clifford algebra are normalized by $Tr(\gamma_\alpha\gamma_\beta) = 4\ o_{\alpha\beta}$ and $Tr(\sigma_{\alpha\beta}\sigma^{\gamma\delta}) = 8\delta^\gamma_{[\alpha}\delta^\delta_{\beta]}$, where $[\alpha\beta] = \frac{1}{2}(\alpha\beta - \beta\alpha)$ denotes the antisymmetrization of indices. In $n = 4$ dimensions, they constitute a *minimal* representation of the corresponding Clifford algebra. An extension to n dimensions was discussed by Brauer and Weyl (1935) as well as Sáenz and Wigner (1967).

The Lorentz generators $\sigma_{\alpha\beta}$ and the totally antisymmetric product γ_5 of Dirac matrices fulfill

$$[\gamma_\alpha, \sigma_{\beta\gamma}] = 2i(o_{\alpha\beta}\gamma_\gamma - o_{\alpha\gamma}\gamma_\beta),$$

$$\{\gamma_\alpha, \sigma_{\beta\gamma}\} = 2i\gamma_{[\alpha}\gamma_\beta\gamma_{\gamma]} = 2\eta_{\alpha\beta\gamma\delta}\gamma^\delta\gamma_5 \tag{2}$$

$$\{\gamma_5, \gamma_\alpha\} = 0, \qquad \gamma_5\gamma_5 = +\ \mathbf{1}_4 \tag{3}$$

Following Schrödinger (1932) (cf. Mielke, 1981a; Mielke, 1987; Hehl *et al.*, 1991b), the constant γ_α matrices can be converted into the Clifford algebra-valued one-form

$$\gamma := \gamma_\alpha \vartheta^\alpha, \qquad \text{where} \quad \vartheta^\alpha := e_i{}^\alpha\, dx^i \quad \text{is a basis for the coframes} \tag{4}$$

For Clifford comodules, see Oziewicz (1998).

The contraction operator acting on p-forms from the left is defined as

$$\check{\gamma} := \gamma^\beta e_\beta = \gamma^\beta e^i{}_\beta \partial_i \qquad \text{with} \quad \check{\gamma} \rfloor \gamma = 4 \cdot \mathbf{1}_4 \tag{5}$$

It generalizes the usual Feynman "dagger" convention $\not{A} := \gamma^\alpha e_\alpha \rfloor A$ of particle physics (Bjorken and Drell, 1964) for one-forms A.

The frame could be regarded as a *truncated* tangent vector of a *Clifford manifold* **C** for which the multivector $\mathbf{X} = \mathbf{x}^\alpha \gamma_\alpha + x^{\beta\gamma}\sigma_{\beta\gamma} + x^{\mu\nu\rho}\gamma_{[\mu}\gamma_\nu\gamma_{\rho]} + \ldots$ represents the hierachy of point, loop, 2-loop (brane), etc., histories in a new *infinite-dimensional* spacetime approach (Castro, 2000) to quantum gravity.

2.1. Dual Forms and Chiral Transformations

Here, we restrict ourselves to a topologically trivial frame bundle where $\eta = (1/4!)\eta_{\alpha\beta\gamma\delta}\, \vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta$ is the volume four-form with the normalization $\eta_{\hat{0}\hat{1}\hat{2}\hat{3}} = +1$, $\eta_\alpha := e_\alpha \rfloor \eta = {}^*\vartheta_\alpha$ is the coframe "density," and * is the Hodge dual. On an 4-dimensional manifold with metric index s, the Hodge dual of p-forms is almost involutive: $^{**}\alpha = (-1)^{p(4-p)+s}\alpha$. For spacetimes where $s = 1$ holds, it induces an *almost complex structure* (Brans, 1975).

Our *Hodge dual* * of exterior forms is defined such that the normalization ${}^*(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta) = \eta^{\alpha\beta\gamma\delta}$ holds. In order to complete the η-basis for forms, we define $\eta_{\alpha\beta} := e_\beta \rfloor \eta_\alpha$, $\eta_{\alpha\beta\gamma} := e_\gamma \rfloor \eta_{\alpha\beta}$, $\eta_{\alpha\beta\gamma\delta} := e_\delta \rfloor \eta_{\alpha\beta\gamma}$ with the aid of the interior product $\rfloor$.

The Hodge dual of the basis Clifform leads to the associated three-form

$${}^*\gamma = \gamma^\alpha \eta_\alpha = \frac{i}{6}\gamma_5 \gamma \wedge \gamma \wedge \gamma \tag{6}$$

whereas for the *Lie* (or right) $\sigma^\star_{\alpha\beta} := \frac{1}{2}\sigma^{\gamma\delta}\eta_{\alpha\beta\gamma\delta} = i\gamma_5\sigma_{\alpha\beta}$, we find the associated two-forms

$$\sigma := \frac{1}{2}\sigma_{\alpha\beta}\vartheta^\alpha \wedge \vartheta^\beta = \frac{i}{2}\gamma \wedge \gamma, \qquad {}^*\sigma = \frac{1}{2}\sigma_{\alpha\beta}\eta^{\alpha\beta} =: \sigma^\star = i\gamma_5\sigma \tag{7}$$

In four dimensions, the Hodge dual applied to two-forms is conformally invariant (Atiyah *et al.*, 1978) under the Weyl rescaling $\gamma \to \tilde{\gamma} = e^\varphi \gamma$, where φ is the the *dilaton* field. Conversely, an initially metric-free *involutive* star operation # on *arbitrary* two-forms allows one to *reconstruct* (Dray *et al.*, 1989; Harnett, 1991) a spacetime metric h which is conformally related to g.

Note that in orthonormal frames the Hodge dual * and the Lie dual * are identical operations for σ (Mielke *et al.*, 1996a). Moreover, this allows one to *reconstruct* the Hodge dual and therefore a conformal equivalence class of spacetime metrics from the Lie dual as defined by Kähler (cf. Trautman, 1999): For the metric-free two-form σ, we can built the Lie dual $\sigma^\star$ solely by multiplication with γ_5, which here is regarded as just an anticommuting element of the Clifford algebra. This Lie dual is antiinvolutive:

$$\sigma^\star := i\gamma_5\sigma, \qquad \sigma^{\star\star} = i^2\gamma_5^2\sigma = -\sigma \tag{8}$$

Moreover, due to (2), we have the metric-free Clifform relation

$$[\gamma, \sigma^\star] = \gamma \wedge i\gamma_5\sigma - i\gamma_5\sigma \wedge \gamma = -i\gamma_5(\gamma \wedge \sigma + \sigma \wedge \gamma) = 2\gamma_5\gamma \wedge \gamma \wedge \gamma \tag{9}$$

Acoording to (6), we can identify this with $12i{}^*\gamma$ and therefore have recovered the Hodge dual for the basis of Clifforms (Harnett, 1992). This allows us to identify γ_5 with the the zero-form $\gamma_5 := (i/4!)^* (\gamma \wedge \gamma \wedge \gamma \wedge \gamma)$.

In a recent paper (Obukhov and Hehl, 1999) a related reconstruction has been discussed on the basis of older work, but specialized to the Faraday two-form $F = dA$. This is not compulsory, as we have just demonstrated; also, it could be applied, for example, to the Kalb–Ramond two-form B arising in the low-energy limit of strings. Moreover, in view of quantum-field-theoretic problems such as conformal anomalies (Deser and Schwimmer, 1993) and the perturbative nonrenormalizability ('t Hooft, 1975) of gravitationally coupled fields, we *cannot* subscribe to a physical interpretation of the metric as arising from a Maxwell-like classical "æther."

For Minkowski signature, the Hodge dual satisfies $** = -1$, therefore often i^* is used in field theory in order to have an *involutive* duality operator. We will encounter also the *self*- or *anti-self-dual* combination

$$\sigma_\pm := (\sigma \pm i{}^*\sigma)/2 = \tfrac{1}{2}(1 \mp \gamma_5)\sigma \qquad \text{with} \quad i{}^*\sigma_\pm = \pm\sigma_\pm \tag{10}$$

which is originally due to Debever (1964) and Brans (1975), but at times is referred to as the *Plebański* (1975, 1977) *two-form*. Our Clifford representation involves explicitly the *chirality projector* $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ obeying $P_\pm P_\pm = P_\pm$.

This decomposition is invariant under the *chiral transformation*

$$\gamma \to \gamma^\theta = e^{i\gamma^5\theta}\gamma e^{-i\gamma^5\theta} \tag{11}$$

of the coframe, where θ denotes the so-called 'theta angle' ('t Hooft, 1991).

2.2. Riemann–Cartan spacetime: Clifford Algebra-Valued Torsion and Curvature

The Riemann–Cartan (RC) geometry (Cartan, 1924) as a basis for the Einstein–Cartan (EC) theory of gravity has been reviewed by Hehl *et al.*

(1976). In terms of the Clifford algebra-valued *connection* $\Gamma := \frac{1}{4} i \Gamma^{\alpha\beta}\sigma_{\alpha\beta}$, the $\overline{SO}_o(1, 3) \cong SL(2, C)$-covariant exterior derivative $D = d + [\Gamma, \;]$ employs the algebra-valued *form commutator* $[\Psi, \Phi] := \Psi \wedge \Phi - (-1)^{pq}\Phi \wedge \Psi$. Differentiation of the basic variables leads to the Clifford algebra-valued *torsion* and *curvature* two-forms, respectively:

$$\Theta := D\gamma = T^\alpha \gamma_\alpha, \qquad \Omega := d\Gamma + \Gamma \wedge \Gamma = \frac{i}{4} R^{\alpha\beta}\sigma_{\alpha\beta} \tag{12}$$

This compact Clifform formula for the curvature is due to Schrödinger (1932), who thereby anticipated the concept of *gauge* curvature developed much later by Yang and Mills (1954).

In this Clifform approach, the torsion two-form can be irreducibly decomposed into the trace part ${}^{(2)}\Theta := \frac{1}{3}\gamma \wedge T$, the axial torsion ${}^{(3)}\Theta := -\frac{1}{3}{}^*(\gamma \wedge A)$, and the tensor torsion ${}^{(1)}\Theta := \Theta - {}^{(2)}\Theta - {}^{(3)}\Theta$, where the one-forms of the trace and axial vector torsion, respectively, are defined by

$$T := \tfrac{1}{4} Tr(\check{\gamma} \rfloor \Theta) = e_\alpha \rfloor T^\alpha, \qquad A := \tfrac{1}{4} {}^*Tr(\gamma \wedge \Theta) = {}^*(\vartheta_\alpha \wedge T^\alpha) \tag{13}$$

Under a Weyl rescaling, the axial torsion remains invariant, i.e., $\tilde{A} = A$, whereas it picks up a gradient term under a chiral transformation:

$$\tilde{T} = T - 3d\vartheta, \qquad A^\theta = A - id\theta \tag{14}$$

In RC spacetime, the translational and Lorentz-rotational *Chern–Simons terms* read

$$\begin{aligned} C_{\rm TT} &:= \frac{1}{8l^2} Tr(\gamma \wedge \Theta) = -\frac{1}{2l^2} {}^*A, \\ C_{\rm RR} &:= -Tr\left(\Gamma \wedge \Omega - \frac{1}{3} \Gamma \wedge \Gamma \wedge \Gamma\right) \end{aligned} \tag{15}$$

where a fundamental length l is introduced for dimensional reasons. The Clifford algebra approach has the advantage that we can employ the trace in the definition (15), whereas the usual translational generators P_α commute and do not have a nondegenerate Cartan–Killing metric.

The Ricci identity reads

$$DD\Psi = [\Omega, \Psi] \tag{16}$$

whereas the first and second *Bianchi identities* adopt in RC geometry the form

$$D\Theta \equiv [\Omega, \gamma], \qquad D\Omega \equiv 0 \tag{17}$$

respectively (Hehl *et al.*, 1991a).

The *Einstein three-form* can be defined by

$$E := E_\alpha \gamma^\alpha := \tfrac{1}{2} R^{\mu\nu} \wedge \eta_{\mu\nu\lambda}\gamma^\lambda = -i\gamma_5(\Omega \wedge \gamma + \gamma \wedge \Omega) = i[\gamma, \gamma_5 \Omega] \tag{18}$$

In view of the *contracted* Bianchi identities

$$D[\gamma, \Theta] \equiv 2i[\sigma, \Omega], \qquad D[\gamma, \Omega] \equiv [\Theta, \Omega], \qquad DE \equiv i[\Theta, \gamma_5\Omega] \tag{19}$$

the automatic conservation of the Einstein three-form holds only for vanishing torsion, i.e., in Einstein's GR with $\Theta = 0$.

The *Lie dual* of Lorentz algebra-valued forms such as contortion and curvature is defined by

$$K^\star := \eta_{\alpha\beta\gamma} \wedge K^{\beta\gamma}\gamma^\alpha, \qquad \Omega^\star := \frac{i}{8} R_{\alpha\beta}\eta^{\alpha\beta\gamma\delta}\sigma_{\gamma\delta} = -\frac{1}{4} R^{\alpha\beta}\gamma_5\sigma_{\alpha\beta} = i\gamma_5\Omega \tag{20}$$

and satisfies $D\Omega^\star \equiv 0$.

We will also employ the self- or anti-self-dual torsion and curvature two-forms

$$\Theta^\pm := \tfrac{1}{2}(\Theta \pm {}^*\Theta), \qquad \Omega^\pm := \tfrac{1}{2}(\Omega \pm {}^*\Omega), \qquad \Omega^{(\pm)} := \tfrac{1}{2}(\Omega \pm \Omega^*) \tag{21}$$

in terms of the Hodge or Lie dual, respectively.

3. YANG–MILLS EQUATION

The prototype gauge theory is that of Yang and Mills formulated originally (Yang and Mills, 1954; Mills, 1989) for the Lie group $G = SU(2)$. Quite generally (Mielke, 1987), for a Lie-algebra-valued one-form $A := A_i^J \lambda_J \, dx^i$ of a non-Abelian gauge theory with Lie generators λ_J, the Yang–Mills field strength and its self- and anti-self-dual parts are given by

$$F := dA + A \wedge A, \qquad F^\pm := \tfrac{1}{2}(F \pm i{}^*F) \tag{22}$$

The corresponding Chern–Simons term

$$C := Tr(A \wedge F - \tfrac{1}{3} A \wedge A \wedge A), \qquad dC = Tr(F \wedge F) \tag{23}$$

of a non-Abelian gauge theory yields the *Pontrjagin four-form dC* as boundary term which is important for topological reasons.

From the Yang–Mills Lagrangian with topological term

$$L_{\rm YM} := -\frac{1}{2}\,Tr(F \wedge {}^*F) + \frac{\theta}{2}\,dC$$
$$= \frac{\theta - i}{2}\,Tr(F^+ \wedge F^+) + \frac{\theta + i}{2}\,Tr(F^- \wedge F^-) \tag{24}$$

we find the *Yang–Mills eguations*

$$D^*F = J, \qquad DF \equiv 0 \tag{25}$$

In the Abelian case, i.e., Maxwell's theory with $G = U(1)$, the covariant derivative $D := d + A\wedge$ reduces to exterior derivatives, as $F = dA$, e.g., also in RC spacetime or more general geometries (Hehl *et al.*, 1995).

3.1 *CP* Violation?

In contrast to the two-spinor approach (Plebański, 1975, 1977), our Clifform formulation embodies not only the *connected component* $\overline{SO_o}$ (1, 3) of the Lorentz group, but *manifestly* also the operators of *reflections* of the *full Lorentz group*, including parity P, an (antiunitary) time reflection T, and the total reflection J. Einstein's theory is invariant under all reflections; after coupling it to Maxwell's theory of electromagnetism, it is also CPT invariant, where C denotes the charge conjugation. From the neutral kaon system, the tightest bound on CPT violation is now 2×10^{-18}.

On the other hand, for gravitational interactions, it is not known experimentally if all reflections are respected or if there exist, for instance, only right- (or left-) handed gravitons (Mielke *et al.*, 1999). In any case, the occurrence of the gravitational boundary terms $dC_{\rm RR}$ and $dC_{\rm TT}$ in the action would violate parity P and, depending on the value of the theta angle, also CP, similarly as the topological term $dC = F \wedge F$ in Table I.

4. DIRAC EQUATION

A Dirac field is a bispinor-valued zero-form ψ for which $\overline{\psi} := \psi^\dagger\gamma_0$ denotes the Dirac adjoint and $D\psi := d\psi + \Gamma \wedge \psi$ is the exterior covariant

Table I. Clifford Representation of Discrete Operators

Discrete symmetry	Operator	Pontrjagin term $F \wedge F$	$iF \wedge F$
Parity	$P = \gamma^{\hat 0}$	-1	-1
Time reversal	$T = i\gamma^{\hat 1}\gamma^{\hat 3}$	-1	$+1$
Charge conjugation	$C = i\gamma^{\hat 2}\gamma^{\hat 0}$	$+1$	-1
CP	$CP = i\gamma^{\hat 2}$	-1	$+1$
CPT	$CPT = \gamma^{\hat 1}\gamma^{\hat 2}\gamma^{\hat 3}$	$+1$	$+1$

derivative. The Dirac Lagrangian is given by the manifestly *Hermitian* four-form

$$L_D = L(\gamma, \psi, D\psi) = \frac{i}{2}\{\overline{\psi}{}^*\gamma \wedge D\psi + \overline{D\psi} \wedge {}^*\gamma\psi\} + m\overline{\psi}\psi\eta \qquad (26)$$

Since $L_D = \overline{L}_D = L_D^\dagger$ even in an unholonomic frame, it provides us automatically with the *Hermitian* charge current $j = e\overline{\psi}{}^*\gamma\psi$. For the mass term, we henceforth will use the short-hand notation ${}^*m = m\eta$.

The Dirac equation and its adjoint are obtained by varying L_D independently with respect to $\overline{\psi}$ and ψ:

$$i{}^*\gamma \wedge D\psi + {}^*m\psi - \frac{i}{2}(D{}^*\gamma)\psi = 0$$

$$i\overline{D\psi} \wedge {}^*\gamma + {}^*m\overline{\psi} + \frac{i}{2}\overline{\psi}D{}^*\gamma = 0 \qquad (27)$$

If we use the properties of the Hodge dual, the term $D{}^*\gamma = {}^*\gamma \wedge (e_\beta \rfloor T^\beta)$ turns out to be proportional to the vector torsion one-form $T := e_\beta \rfloor T^\beta$. Consequently, the Dirac equation adopts the equivalent form

$$i{}^*\gamma \wedge \left(D + \frac{i}{4}m\gamma - \frac{1}{2}T\right)\psi = 0 \qquad (28)$$

In view of $L_D \cong 0$ 'on shell', the canonical energy-momentum three-form of the Dirac field reads

$$\Sigma_\alpha \cong \frac{i}{2}\{\overline{\psi}{}^*\gamma \wedge D_\alpha\psi - \overline{D_\alpha\psi} \wedge {}^*\gamma\psi\}, \qquad D_\alpha := e_\alpha \rfloor D \qquad (29)$$

The spin current of the Dirac field is given by the Hermitian three-form

$$\tau := \frac{i}{4}\sigma^{\alpha\beta}\frac{\partial L_D}{\partial \Gamma^{\alpha\beta}} = \frac{i}{4}\sigma^{\alpha\beta}\tau_{\alpha\beta}$$

$$= \frac{i}{32}\overline{\psi}({}^*\gamma\sigma_{\alpha\beta} + \sigma_{\alpha\beta}{}^*\gamma)\psi\sigma^{\alpha\beta} = \frac{1}{8}\gamma_5\sigma_{\gamma\delta}\overline{\psi}\gamma_5\gamma^\delta\psi\eta^\gamma \qquad (30)$$

(Mielke, 1987, Eq. (4.2.13)). This implies that the components $\tau_{\alpha\beta\gamma} := e_\gamma \rfloor {}^*\tau_{\alpha\beta} = \tau_{[\alpha\beta\gamma]}$ of the spin current are *totally antisymmetric*.

4.1. Nonlinear Heisenberg Spinor Equation versus Dynamical Axion Coupling

In order to separate out in the Dirac equation the purely Riemannian piece from torsion terms, we decompose the Riemann–Cartan connection

$\Gamma = \Gamma^{\{\}} - K$ into the Riemannian (or Christoffel) connection $\Gamma^{\{\}}$ and the *contortion* one-form K. By inserting this decomposition into Eq. (26), the Dirac Lagrangian splits into a Riemannian and a spin-contortion piece (Hehl *et al.*, 1991b; Mielke *et al.*, 1996a):

$$\begin{aligned} L_{\mathrm{D}} &= L(\gamma, \psi, D^{\{\}}\psi) - \frac{i}{2}\overline{\psi}({}^*\gamma \wedge K - K \wedge {}^*\gamma)\psi \\ &= L(\gamma, \psi, D^{\{\}}\psi) - \frac{1}{4} A \wedge i_5 \end{aligned} \tag{31}$$

where $j_5 := \overline{\psi} \wedge \gamma_5{}^*\gamma\psi$ is the familiar *axial current* of Dirac fields.

The Dirac equation obtained by varying the decomposed Lagrangian (31) reads

$$i{}^*\gamma \wedge \left[D^{\{\}} + \frac{i}{4} m\gamma - \frac{i}{4} A\gamma_5\right]\psi = 0 \tag{32}$$

Note that for a Riemannian covariant derivative, we have $D^{\{\}}\gamma = 0$. Hence, in a Riemann–Cartan spacetime, a Dirac spinor only feels the *axial torsion* one-form $A := \frac{1}{4} Tr(\tilde{\gamma}\rfloor{}^*\Theta) = \frac{1}{4} Tr{}^*(\gamma \wedge \Theta)$.

In the Einstein–Cartan theory, the torsion fulfills (38) (Hehl *et al.*, 1976). Since this implies $A = -(l^2/2)\overline{\psi}\gamma_5\gamma\psi$, the axial torsion term can be removed in (32), thus leading to the effectively *nonlinear* Dirac equation (Weyl, 1929; Hehl and Datta, 1971; Mielke, 1981c)

$$i{}^*\gamma \wedge D^{\{\}}\psi + {}^*m\psi + \frac{l^2}{24}\overline{\psi}\gamma_5\gamma\psi \wedge \sigma \wedge \gamma\psi = 0 \tag{33}$$

in Riemannian spacetime.

However, it is well known that this *nonlinear* Heisenberg–Pauli–Weyl equation with its ψ^3-type self-interaction (Mielke, 1987) is perturbatively nonrenormalizable (Fauser, 1998). Moreover, in quantum field theory (QFT), there arises in RC spacetime the *axial anomaly*

$$\langle dj_5\rangle = 2im\langle\overline{\psi}\gamma_5\psi\rangle + \frac{1}{24\pi^2}\left[Tr(\Omega^{\{\}} \wedge \Omega^{\{\}}) - \frac{1}{4} dA \wedge dA\right] \tag{34}$$

This result (Mielke and Kreimer, 1998, 1999), which can easily be transferred to the chiral current $j_\pm$, is based on the Pauli–Villars regularization scheme. It deviates from the heat kernel method (Obukhov, 1983; Obukhov *et al.*, 1997), which leads to partially *divergent* terms. Thus, only the term $dA \wedge dA$ arises in the axial anomaly (Mielke and Kreimer; 1998, 1999), but *not* the Nieh–Yan (NY) type term $d{}^*A = -2l^2 dC_{\mathrm{TT}}$ as has been recently claimed (Chandia and Zanelli, 1998). A consistent way to avoid (Kreimer and Mielke,

2000) regularization problems is to impose the transversallity condition $d{*}A = 0$ on the axial torsion, which just implies the vanishing of the NY term.

In *effective string models* (Duff, 1995), the theta angle of the topological boundary term θdC is liberated to a pseudoscalar, the Kalb–Ramond (KR) *axion*. Then it may serve as a pseudoscalar potential for the axial torsion one-form via $A = id\theta$, as can be inferred from the chiral transformation (14).

Besides the Chern–Simons terms and the dilaton coupling, it enters in the Kalb–Ramond three-form $H := e^{\varphi/f_\varphi}\, {*}d\theta - C - l^2 C_{\mathrm{TT}} - C^{\{\}}_{\mathrm{RR}}$. Since $dA = idd\theta \equiv 0$ due to the Poincaré lemma, the Pontrjagin-type term in (34) for the axial torsion gets removed. Anomaly cancellation for strings then requires the Bianchi identity

$$dH = Tr(\Omega^{\{\}} \wedge \Omega^{\{\}} - F \wedge F) \tag{35}$$

Contrary to Heisenberg's nonlinear four-fermion contact interaction, the axial torsion part $A \wedge {*}A = -d\theta \wedge {*}d\theta$ in the decomposition of the Einstein–Cartan Lagrangian (36) has become *dynamical* (Duncan *et al.*, 1992) due to the anomaly. This makes quantum electrodynamics (QED) in RC spacetimes equivalent to QED on a *torsionless* spacetime geometry coupled to the axion. Since the geometry couples, via the topological Pontrjagin term, back to the effective axion field equation (35), black holes may get restyled by "axion hair" of odd parity.

5. EINSTEIN–CARTAN EQUATIONS

For the Einstein–Cartan (EC) theory, the Lagrangian reads in Clifforms

$$V_{\mathrm{EC}} = \frac{i}{2l^2}\, Tr(\Omega \wedge {*}\sigma) = -\frac{1}{2l^2}\, Tr(\Omega \wedge \gamma_5 \sigma) \tag{36}$$

(Weyl, 1929; Trautman, 1973). Comparing with the Poincaré gauge approach (Hehl *et al.*, 1976, 1995), the translational field momentum $H_{\mathrm{EC}} := -\gamma^\alpha\, \partial V_{\mathrm{EC}}/\partial T^\alpha$ vanishes, whereas the Lorentz field momentum $H_{\mathrm{EC}} := -\frac{1}{4} i \sigma^{\alpha\beta}\, \partial V_{\mathrm{EC}}/\partial R^{\alpha\beta} = -(1/4l^2)\gamma_5\sigma$ turns out to be proportional to the dual of the "unit" curvature two-form.

Thus the first field equation reduces to

$$-E = i[\gamma_5 \Omega, \gamma] = l^2 \Sigma \tag{37}$$

where E is the Einstein three-form (18), $\Sigma := \gamma^\alpha(\partial L/\partial \vartheta^\alpha) = \gamma^\alpha \Sigma_\alpha$ the canonical *energy-momentum* current, and l^2 the Planck length squared as gravitational coupling constant in natural units where $c = \hbar = 1$. The second field equation simplifies to *Cartan's algebraic relation* between torsion and spin,

$$[\Theta, \gamma] = 8il^2\gamma_5\tau \tag{38}$$

If we introduce the spin-energy potential two-form $\mu = \gamma^\alpha\mu_\alpha = \frac{1}{4}\gamma \wedge \overline{\psi}\gamma\gamma_5\psi$ via $\tau =: \frac{1}{8}\gamma_5[\mu, \gamma]$ or $\tau_{\alpha\beta} = \vartheta_{[\alpha} \wedge \mu_{\beta]}$ in components, then we can resolve (38) explicitly for the torsion $\Theta = \frac{1}{2}i\, l^2\gamma_5(\mu - \frac{1}{4}\gamma \wedge \breve{\gamma}\rfloor\mu)$.

The Hilbert–Einstein Lagrangian of GR is recovered for vanishing spin, or via Belinfante symmetrization of the canonical energy-momentum tensor Σ (Mielke *et al.*, 1989; Hehl *et al.*, 1999). The teleparallelism equivalent of GR has been analyzed for *knot states* (Mielke, 1998, 1999) as solutions of the Hamiltonian constraints (Mielke, 1992).

5.1. Chiral Gravity and Ashtekar Reformulation

The Ashtekar reformulation (1986, 1991) of GR can be obtained via a canonical transformation induced by a boundary term which does *not* change the local physics of the EC action. The exact form derived from the *translational* Chern–Simons term

$$C_{\rm TT} := \frac{1}{8l^2}\, Tr(\gamma \wedge \Theta) \tag{39}$$

which is also known as the *Nieh–Yan (1982) term*

$$dC_{\rm TT} = \frac{1}{8l^2}\, Tr(\Theta \wedge \Theta - \gamma \wedge [\Omega, \gamma]) = \frac{1}{8l^2}\, Tr(\Theta \wedge \Theta - 4i\Omega \wedge \sigma) \tag{40}$$

is instrumental here and also in the chiral transition of the fermionic matter Lagrangians. Since this term induces parity-violating pieces (Mielke *et al.*, 1999), one has to add to it the Lagrangian with the imaginary unit *i* as factor in order to preserve the more stringent *CP* invariance of quantum field theory. This boundary term represents the *generating function* (Mielke, 1990b, 1992) of our canonical transformation to variables involving the *self-dual* or *anti-self-dual* connection. Then the complex EC Lagrangian reads

$$\overset{(\pm)}{V}_{\rm EC} := V_{\rm EC} \pm i dC_{\rm TT} = \pm\frac{1}{2l^2}\, Tr\left\{(1 \mp \gamma_5)\Omega \wedge \sigma + \frac{i}{4}\,\Theta \wedge \Theta\right\}$$
$$= \pm\frac{1}{l^2}\, Tr\left(\Omega_\pm \wedge \sigma_\pm + \frac{i}{8}\,\Theta \wedge \Theta\right) \tag{41}$$

This leads to the following identifications of the new field momenta:

$$\Theta = \pm\, il^2\overset{(\pm)}{H}_{\rm EC}, \qquad \sigma_\pm := P_\mp\sigma = \pm 2l^2\overset{(\pm)}{H}_{\rm EC}, \qquad \Omega_\pm = P_\mp\Omega$$

Likewise, the transition to the chiral Dirac Lagrangian can be achieved by adding the translational Chern–Simons term as generating function, but now evaluated "on shell" via (38). Then we see that is is proportional to the *axial current*

$$C_{\mathrm{TT}} \cong \tfrac{1}{4}\overline{\psi}\, \gamma_5{}^*\gamma\psi = \tfrac{1}{4}j_5 \tag{42}$$

In this reformulation, a quadratic torsion term occurs in Eq. (41). According to (42), for the Einstein–Cartan theory coupled to the Dirac field (ECD), the induced torsion can be expressed solely in terms of the *axial* torsion ${}^{(3)}\Theta$ (Hehl *et al.*, 1991b). Therefore, we have in this case ${}^{(1)}\Theta = {}^{(2)}\Theta = 0$. This immediately implies that the quadratic torsion term vanishes for the Dirac field coupled to EC gravity due to the geometric identity

$$\Theta \wedge \Theta \equiv {}^{(1)}\Theta \wedge {}^{(1)}\Theta + 2{}^{(2)}\Theta \wedge {}^{(3)}\Theta = 0 \tag{43}$$

The same happens for supergravity (coupled Einstein–Cartan–Rarita–Schwinger fields) (see Mielke *et al.*, 1996a, b; Mielke and Macías, 1999, for details).

6. SIMPLE SUPERGRAVITY

The simplest consistent coupling of a Rarita–Schwinger (RS)-type spin-3/2 field Ψ to gravity is *supergravity* (Nieuwenhuizen, 1981; Freedman, 1993) with one supersymmetry generator, i.e., $N = 1$.

The corresponding Hermitian Lagrangian four-form reads

$$L_{\mathrm{Sugra}} = V_{\mathrm{EC}} + V_{\mathrm{RS}} = V_{\mathrm{EC}} - \tfrac{1}{2}(\overline{\Psi} \wedge \gamma_5\gamma \wedge D\Psi - \overline{D\Psi} \wedge \gamma_5\gamma \wedge \Psi) \tag{44}$$

where the Rarita–Schwinger field $\Psi := \Psi_\alpha\vartheta^\alpha$ is a *Majorana spinor*-valued one-form (Kaku, 1993).

With our Clifford definition (12) for the torsion, we find for the *Rarita–Schwinger equation* (Urrutia and Vergara, 1991)

$$\gamma \wedge D\Psi - \tfrac{1}{2}\Theta \wedge \Psi = 0 \tag{45}$$

For the coupled Einstein–Cartan–Rarita–Schwinger Lagrangian, the first field equation reads

$$i\gamma_5(\Omega \wedge \gamma + \gamma \wedge \Omega) = \frac{l^2}{2}\, \gamma^\alpha(\overline{\Psi} \wedge \gamma_5\gamma_\alpha D\Psi + \overline{D\Psi} \wedge \gamma_5\gamma_\alpha\Psi) \tag{46}$$

Because of the Cartan-type relation

$$\Theta = T^\alpha\gamma_\alpha = -\frac{i}{2}\, l^2\overline{\Psi} \wedge \gamma^\alpha\Psi\gamma_\alpha \tag{47}$$

we find

$$\Theta \wedge \Psi = -\frac{i}{2}\, l^2 \overline{\Psi} \wedge \gamma^\alpha \Psi \wedge \gamma_\alpha \Psi = 0 \tag{48}$$

as can be easily shown by means of a commutation of the one-form $\gamma^\alpha \Psi$ among itself, i.e., a Fierz reordering (Nieuwenhuizen, 1981). Thus, in sharp contrast to the Dirac case, for the supergravity coupling of the Rarita–Schwinger field, torsion is an *auxiliary* field which does *not* induce a nonlinear term in (45).

Covariant exterior differentiation of the Rarita–Schwinger equation (45) yields the *integrability condition*:

$$\begin{aligned} D(\gamma \wedge D\Psi - \tfrac{1}{2}\Theta \wedge \Psi) &= \Theta \wedge D\Psi - \gamma \wedge DD\Psi - \tfrac{1}{2}[\Omega, \gamma] \wedge \Psi \\ &\quad - \tfrac{1}{2}\Theta \wedge D\Psi \\ &= \tfrac{1}{2}\Theta \wedge D\Psi - \tfrac{1}{2}(\Omega \wedge \gamma + \gamma \wedge \Omega) \wedge \Psi = 0 \end{aligned} \tag{49}$$

It is a remarkable fact of supergravity also in higher dimensions (Bañados *et al.*, 1996) that the integrability condition (49) for the fermionic fields are the *bosonic* equations (46) and (47). This implies the *gauge invariance* of simple supergravity under invariance of L_{Sugra} under such local *supersymmetric transformations* (SUSY) (Deser and Zumino, 1976), a fact which can be more naturally demonstrated by using the Clifford algebra-valued coframe and connection (Mielke *et al.*, 1996a).

6.1. Chiral Supergravity Induced via a Translational Chern–Simons Term

In order to give the supergravity Lagrangian (44) its chiral form, an analysis similar to the one given in the previous section should be performed, focusing on the more fundamental point of view of the generating function (Mielke *et al.*, 1996a, b).

In our elegant *"Clifform"* approach, we note that "on shell," i.e., after using the Cartan relation (47), the translational Chern–Simons term (39) is given by

$$C_{\text{TT}} \cong \frac{i}{4}\, \overline{\Psi} \wedge \gamma \wedge \Psi = \frac{1}{4} J_5 \tag{50}$$

which is proportional to the *axial current* $J_5 := i\overline{\Psi} \wedge \gamma \wedge \Psi$ of the Rarita–Schwinger field.

Similarly as in the Dirac case, the chiral version of the Rarita–Schwinger Lagrangian is obtained (Mielke *et al.*, 1996b; Mielke and Macías, 1999) by adding the boundary term dC_{TT} multiplied by the imaginary unit:

$$L_{\mathrm{RS}\pm} := \tfrac{1}{2}L_{\mathrm{RS}} \mp idC_{\mathrm{TT}}$$
$$= \pm\tfrac{1}{2}(\overline{D\Psi} \wedge P_{\pm}\gamma \wedge \Psi + \overline{\Psi} \wedge P_{\mp}\gamma \wedge D\Psi) \tag{51}$$

In full, the *chiral* supergravity theory can be related to simple supergravity through the identity

$$L^{\mathrm{chiral}}_{\mathrm{Sugra}} := \tfrac{1}{2}\overset{(\pm)}{V}_{\mathrm{EC}} + L_{\mathrm{RS}\pm} = \tfrac{1}{2}(L_{\mathrm{Sugra}} \mp idC_{\mathrm{TT}}) \tag{52}$$

7. SEIBERG–WITTEN EQUATIONS

The *duality* of electric and magnetic fields in Maxwell's theory was already known to Von Laue (Sommerfeld, 1910). In 1925 the symmetry of *duality rotations* was realized by Rainich (1925) and developed further in *geometrodynamics* by Misner and Wheeler (1957, 1987). Montonen and Olive (1978) observed in the context of magnetic monopoles that this generates also a duality of the strong–weak coupling regime of gauge fields. This so-called *S-duality* nowadays plays a predominant role in M-theory (Duff, 1995; Witten, 1998). These ideas were taken up by Seiberg and Witten (1994) because they may have important consequences for quark confinement and the Higgs field (Yung, 1999). From a mathematical perspective, the Donaldson invariants of four-dimensional manifolds should be calculable in terms of classical solutions of a system of gauge equations coupled to spinors.

Let us start from the Seiberg–Witten (SW) Lagrangian (Jost *et al.*, 1995),

$$\begin{aligned} L_{\mathrm{SW}} &= \frac{1}{2}\,\overline{D_{\pm}\psi} \wedge {}^*D_{\pm}\psi \mp i\left(F^{\pm} - \frac{1}{2}\,\overline{\psi}\sigma_{\pm}\psi\right)^2 \\ &= \mp i\,Tr(F^{\pm} \wedge F^{\pm}) + \frac{1}{2}\,\overline{D_{\pm}\psi} \wedge {}^*D_{\pm}\psi \\ &\quad \pm i\overline{\psi}\sigma_{\pm}\psi \wedge F^{\pm} \mp \frac{i}{4}\,\overline{\psi}\sigma_{\pm}\psi \wedge \overline{\psi}\sigma_{\pm}\psi \end{aligned} \tag{53}$$

where we suppressed the decomposition $\psi = \psi_L + \psi_R = P_{-}\psi + P_{+}\psi$ of the Dirac spinors into the left- and right-handed pieces. The gauge part of the SW Lagrangian corresponds to the chiral decomposition (24) induced by $\theta = \mp i$.

It can be regarded as a self- or anti-self-dual Yang–Mills Lagrangian coupled to the convective and polarization Lagrangian resulting from a Gordon decomposition of a Dirac field (Hehl *et al.*, 1999). The squared term of the polarization two-form $P := (i/2)\overline{\psi}\sigma_{\pm}\psi$, as is typical for a four-fermion type self-interaction, plays the role of an effective mass term.

The variation with respect to $\overline{\psi}$ and A leads to the *convective* spinor equation

$$D_{\pm} \, {}^*D_{\pm}\psi \mp i(F^{\pm} - \tfrac{1}{2}\overline{\psi}\sigma_{\pm}\psi) \wedge \sigma_{\pm}\psi = 0 \tag{54}$$

coupled to the Yang–Mills-type equation

$$D_{\pm}\left(F^{\pm} - \frac{1}{2}\,\overline{\psi}\sigma_{\pm}\psi\right) = \mp\frac{i}{2}\,\overline{\psi}{}^*D_{\pm}\psi = \mp\frac{1}{2}\,\overline{\psi}{}^*\gamma \wedge {}^*(i{}^*\gamma \wedge D_{\pm}\psi) \tag{55}$$

Solutions (Saçlioglu, 1999) of this system necessarily satisfy the *Seiberg–Witten equations*

$$i{}^*\gamma \wedge D_{\pm}\psi = 0 \qquad \text{and} \qquad F^{\pm} = \tfrac{1}{2}\overline{\psi}\sigma_{\pm}\psi \tag{56}$$

which *linearize* the system (54), (55) and, for Euclidean signature, also *minimize* the Lagrangian (53). It is interesting to note that the algebraic SW relation for the gauge field strength $F^{\pm}$ resembles the modified (double) duality Ansatz (Mielke, 1981b, 1984a, b; Baekler *et al.*, 1982; 1986).

$$\Omega^{\pm} = \frac{1}{2l^2}\,\sigma_{\pm} \tag{57}$$

used for the RC curvature in the Poincaré gauge theory of gravity. Its solutions are known (Baekler, 1981; Baekler and Mielke, 1986) to be of the anti-de Sitter (AdS) type. Today it is advocated to use the effective $D = 11$ supergravity resulting from M-theory after compactification to AdS space (Witten, 1998) as a calculational means ("analog computer") (Ne'eman, 2000) for the strong coupling regime of *quark confinement* in QCD.

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